Action principle for nonlinear parametric quantization of gravity

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Abstract

The derivation of the recently proposed nonlinear quantum evolution of gravity from an action principle is considered in this brief note. It is shown to be possible if a set of consistency conditions are satisfied that are analogous to the Dirac relations for the super-Hamiltonian and momenta in classical canonical gravity.

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Recently the quantization of a parameterized theory in finite dimensions with canonical coordinates $q^\mu(t), p_\mu(t)$, $(\mu = 0, 1, \cdots n)$, lapse function $N(t)$ with a parameter time t and Hamiltonian $H(q^\mu, p_\mu, N) = N\mathcal{H}(q^\mu, p_\mu)$ for some $\mathcal{H}$ was considered based on a nonlinear parametric approach [1]. A new quantization scheme was set up by choosing one of the variables, say $\tau := q^0$ and its conjugate momentum $\varpi := p_0$, as classical variables that interact with the remaining quantized variables q^a, $(a = 1, 2, \cdots, n)$ "semi-classically":

$$\imath \frac{\partial \psi}{\partial t} = \hat{H}\psi \tag{1}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left\langle \frac{\partial \hat{H}}{\partial \varpi} \right\rangle, \quad \frac{\mathrm{d}\varpi}{\mathrm{d}t} = -\left\langle \frac{\partial \hat{H}}{\partial \tau} \right\rangle \tag{2}$$

$$\left\langle \hat{\mathcal{H}} \right\rangle = 0 \tag{3}$$

where $\psi = \psi(q^a, t)$ and $\left\langle \hat{O} \right\rangle$ denotes the expectation value of any operator $\hat{O}$. [1] The operators $\hat{\mathcal{H}}$ and $\hat{H} = N\hat{\mathcal{H}}$ are obtained by substituting $p_a \to \hat{p}_a := -\imath \frac{\partial}{\partial q^a}$ into $\mathcal{H}$ respectively (followed by a suitable factor ordering.) The presence of (2) and (3) makes the quantum evolution *nonlinear*. Like many other nonlinear quantum theories previously considered in the literature (e.g. [2]), the system

[1] Units in which $c = \hbar = 16\pi G = 1$ are adopted.

of evolution (1), (2) and (3) can be derived from an action principle where the quantum and classical variables involved are varied. To see this first note that (1) may be replaced by

$$\imath \frac{\partial \psi}{\partial t} = \hat{h}\psi \tag{4}$$

where

$$\hat{h} := \hat{H} - \varpi \dot{\tau}. \tag{5}$$

Since $\hat{h}$ differs from $\hat{H}$ only by a function of t the resulting wavefunctions are related by a time-dependent phase. Clearly equations (4), (2) and (3) arise from extremizing the action

$$S[\psi(q^a, t), \tau(t), \varpi(t), N(t)] := \int \Re \left\langle \psi, \left(\imath \partial_t - \hat{h}\right) \psi \right\rangle \mathrm{d}t \tag{6}$$

with respect to ψ and its conjugate, τ, ϖ and N.

This formulation suggests a parametric quantization of gravity whose formal treatment can be outlined as follows. Start from the Dirac-ADM Hamiltonian for canonical general relativity [3]:

$$H[g_{ij}, p^{ij}, N_\mu] := \int N_\mu \mathcal{H}^\mu(g_{ij}, p^{ij})\, \mathrm{d}^3x \tag{7}$$

in terms of the 3-metric components $g_{ij}(x) = g_{ij}(x^k, t)$ and their conjugate momenta $p^{ij}(x)$, lapse function $N_0(x) = N(x)$, shift functions $N_i(x)$, super-Hamiltonian $\mathcal{H}^0 = \mathcal{H}$ and super-momenta $\mathcal{H}^i$. ($\mu = 0, 1, 2, 3; i, j, k = 1, 2, 3$.) A natural way of isolating the "true" gravitational degrees of freedom is to canonically transform from $g_{ij}(x), p^{ij}(x)$ to a set of four "embedding variables" $\vartheta^\mu(x)$ with conjugate momenta $\varpi_\mu(x)$, ($\mu = 0, 1, 2, 3$) and a set of two unconstrained variables $q^r(x)$ with conjugate momenta $p_r(x)$, ($r = 1, 2$) [4]. This allows us to re-express H as $H[q^r, p_r, \vartheta^\mu, \varpi_\mu, N_\mu]$. The standard interpretation of the former set is that ϑ^0 specifies time slicing whereas ϑ^i set a spatial coordinate condition. For our discussion, the key idea is to treat ϑ^μ as constrained classical variables coupled to the quantized true degrees of freedom carried by q^r via the Hamiltonian operator $\hat{H}[q^r, \hat{p}_r, \vartheta^\mu, \varpi_\mu, N_\mu]$ obtained from H using $p_r \to \hat{p}_r := -\imath \frac{\partial}{\partial q^r}$ with a suitable factor ordering.

Guided by the discussions above we now construct the action integral

$$S[\psi[q^r(x); t), \vartheta^\mu(x), \varpi_\mu(x), N_\mu(x)] := \int \Re \left\langle \psi, \left(\imath \partial_t - \hat{h}\right) \psi \right\rangle \mathrm{d}t \tag{8}$$

where

$$\hat{h}[q^r, \varpi_r, \vartheta^\mu, \dot{\vartheta}^\mu, N_\mu] := \hat{H} - \int \dot{q}^r p_r\, \mathrm{d}^3x. \tag{9}$$

The nonlinear quantum evolution of gravity is generated by varying (8) with respect to the state functional $\psi[q^r(x); t)$ and its conjugate, and the classical variables $\vartheta^\mu(x)$, $\varpi_\mu(x)$ and $N_\mu(x)$. This yields

$$\imath \frac{\partial \psi}{\partial t} = \hat{h}\psi \tag{10}$$

$$\frac{\partial \vartheta^\mu}{\partial t} = \left\langle \frac{\partial \hat{H}}{\partial \varpi_\mu} \right\rangle, \quad \frac{\partial \varpi_\mu}{\partial t} = -\left\langle \frac{\partial \hat{H}}{\partial \vartheta^\mu} \right\rangle \tag{11}$$

$$\left\langle \hat{\mathcal{H}}^\mu \right\rangle = 0. \tag{12}$$

The structure of the above system is similar to that of (4), (2) and (3). However, it is necessary to establish the consistency condition of this system. It follows from (10) and (11) that

$$\frac{\partial}{\partial t}\left\langle \hat{\mathcal{H}}^\mu \right\rangle = \{\hat{\mathcal{H}}^\mu, \hat{H}\} \tag{13}$$

where

$$\{\hat{A}, \hat{B}\} := \int \left(\left\langle \frac{\delta \hat{A}}{\delta \vartheta^\mu(x)} \right\rangle \left\langle \frac{\delta \hat{B}}{\delta \varpi_\mu(x)} \right\rangle - \left\langle \frac{\delta \hat{B}}{\delta \vartheta^\mu(x)} \right\rangle \left\langle \frac{\delta \hat{A}}{\delta \varpi_\mu(x)} \right\rangle \right) \mathrm{d}^3 x - \imath \left\langle [\hat{A}, \hat{B}] \right\rangle \tag{14}$$

for any operators $\hat{A}$ and $\hat{B}$. Provided that a set of embedding variables ϖ_μ and an operator ordering in defining $\hat{\mathcal{H}}^\mu$ can be found so that

$$\{\hat{\mathcal{H}}^\mu, \hat{H}\} = C^\mu_\nu \left\langle \hat{\mathcal{H}}^\nu \right\rangle \tag{15}$$

for some coefficients C^μ_ν (as functions of spacetime points), the consistency of the evolution system (10), (11) and (12) will be satisfied. It is worth noting that the form of (15) is closely analogous to that derived from the "Dirac algebra" in classical canonical gravity. Further progress on the validity of this consistency condition will be reported elsewhere.

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